

RINGS OF INVARIANT MODULE TYPE AND AUTOMORPHISM-INVARIANT MODULES

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Dedicated to T. Y. Lam on his 70 th Birthday

ABSTRACT. A module is called automorphism-invariant if it is invariant under any automorphism of its injective hull. In [Algebras for which every indecomposable right module is invariant in its injective envelope, Pacific J. Math., vol. 31, no. 3 (1969), 655-658] Dickson and Fuller had shown that if R is a finite-dimensional algebra over a field \mathbb{F} with more than two elements then an indecomposable automorphism-invariant right R -module must be quasi-injective. In this paper we show that this result fails to hold if \mathbb{F} is a field with two elements. Dickson and Fuller had further shown that if R is a finite-dimensional algebra over a field \mathbb{F} with more than two elements, then R is of right invariant module type if and only if every indecomposable right R -module is automorphism-invariant. We extend the result of Dickson and Fuller to any right artinian ring. A ring R is said to be of right automorphism-invariant type (in short, RAI-type) if every finitely generated indecomposable right R -module is automorphism-invariant. In this paper we completely characterize an indecomposable right artinian ring of RAI-type.

1. INTRODUCTION

All our rings have identity element and modules are right unital. A right R -module M is called an *automorphism-invariant module* if M is invariant under any automorphism of its injective hull, i.e. for any automorphism σ of $E(M)$, $\sigma(M) \subseteq M$ where $E(M)$ denotes the injective hull of M .

Indecomposable modules M with the property that M is invariant under any automorphism of its injective hull were first studied by Dickson and Fuller in [5] for the particular case of finite-dimensional algebras over fields \mathbb{F} with more than two elements. But for modules over arbitrary rings, study of such a property has been initiated recently by Lee and Zhou in [14]. The dual notion of these modules has been proposed by Singh and Srivastava in [17].

The obvious examples of the class of automorphism-invariant modules are quasi-injective modules and pseudo-injective modules. Recall that a module M is said to be *N -injective* if for every submodule N_1 of the module N , all homomorphisms $N_1 \rightarrow M$ can be extended to homomorphisms $N \rightarrow M$. A right R -module M is *injective* if M is N -injective for every $N \in \text{Mod-}R$. A module M is said to be

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quasi-injective if M is M -injective. A module M is called pseudo-injective if every monomorphism from a submodule of M to M extends to an endomorphism of M .

Thus we have the following hierarchy;

$$\text{injective} \implies \text{quasi-injective} \implies \text{pseudo-injective} \implies \text{automorphism-invariant}$$

It is well known that a quasi-injective module need not be injective. In [18] Teply gave construction of a pseudo-injective module which is not quasi-injective. We do not know yet an example of an automorphism-invariant module which is not pseudo-injective.

Dickson and Fuller [5] studied automorphism-invariant modules in case of finite-dimensional algebras over a field \mathbb{F} with more than two elements. They proved that if R is a finite-dimensional algebra over a field \mathbb{F} with more than two elements then an indecomposable automorphism-invariant right R -module must be quasi-injective. We show that this result fails to hold if \mathbb{F} is a field with two elements. A ring R is said to be of *right invariant module type* if every indecomposable right R -module is quasi-injective. Dickson and Fuller had further shown that if R is a finite-dimensional algebra over a field \mathbb{F} with more than two elements, then R is of right invariant module type if and only if every indecomposable right R -module is automorphism-invariant. We extend the result of Dickson and Fuller to any right artinian ring.

We call a ring R to be of *right automorphism-invariant type* (in short, RAI-type), if every finitely generated indecomposable right R -module is automorphism-invariant. In this paper we study the structure of indecomposable right artinian rings of RAI-type.

Lee and Zhou in [14] asked whether every automorphism-invariant module is pseudo-injective. In this paper we show that the answer is in the affirmative for modules with finite Goldie dimension.

We also prove that a simple right noetherian ring R is a right SI ring if and only if every cyclic singular right R -module is automorphism-invariant.

Before presenting the proofs of these results, let us recall some basic definitions and facts. A module M is said to have finite Goldie (or uniform) dimension if it does not contain an infinite direct sum $\bigoplus_{n \in \mathbb{N}} M_n$ of non-zero submodules.

A module M is called *directly-finite* if M is not isomorphic to a proper summand of itself. Clearly, a module with finite Goldie dimension is directly-finite. A module M is called a *square* if $M \cong X \oplus X$ for some module X ; and a module is called *square-free* if it does not contain a non-zero square.

A module M is said to have the *internal cancellation property* if whenever $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, then $B_1 \cong B_2$. For details on internal cancellation property, the reader is referred to [13]. Now, if an injective module M is directly-finite, then it has internal cancellation property (see [15, Theorem 1.29]).

A module M is said to be *uniserial* if any two submodules of M are comparable with respect to inclusion. A ring R is called a *right uniserial ring* if R_R is a uniserial module. Any direct sum of uniserial modules is called a *serial* module. A ring R is said to be a *right serial ring* if the module R_R is serial. A ring R is called a *serial ring* if R is both left as well as right serial.

If A is an essential submodule of B , then we denote it as $A \subseteq_e B$. For any module M , we define $Z(M) = \{x \in M : \text{ann}_r(x) \subseteq_e R_R\}$. It can be easily checked that $Z(M)$ is a submodule of M . It is called the *singular submodule* of M . If $Z(M) = M$, then M is called a *singular module*. If $Z(M) = 0$, then M is called a *non-singular module*.

Consider the following three conditions on a module M ;

C1: Every submodule of M is essential in a direct summand of M .

C2: Every submodule of M isomorphic to a direct summand of M is itself a direct summand of M .

C3: If N_1 and N_2 are direct summands of M with $N_1 \cap N_2 = 0$ then $N_1 \oplus N_2$ is also a direct summand of M .

A module M is called a *continuous module* if it satisfies conditions C1 and C2. A module M is called *π -injective* (or *quasi-continuous*) if it satisfies conditions C1 and C3. A module M is called a *CS module* (or *extending module*) if it satisfies condition C1.

In general, we have the following implications.

$$\text{Injective} \implies \text{Quasi-injective} \implies \text{Continuous} \implies \pi\text{-injective} \implies \text{CS}$$

The socle of a module M is denoted by $\text{Soc}(M)$. A right R -module M is called *semi-artinian* if for every submodule $N \neq M$, $\text{Soc}(M/N) \neq 0$. A ring R is called right semi-artinian if R_R is semi-artinian. We denote by $J(R)$, the Jacobson radical of a ring R . For any term not defined here, the reader is referred to [9], [11], [12], and [15].

2. BASIC FACTS ABOUT AUTOMORPHISM-INVARIANT MODULES

Lee and Zhou proved the following basic facts about automorphism-invariant modules [14].

- A module M is automorphism-invariant if and only if every isomorphism between any two essential submodules of M extends to an automorphism of M .
- A direct summand of an automorphism-invariant module is automorphism-invariant.
- If for two modules M_1 and M_2 , $M_1 \oplus M_2$ is automorphism-invariant, then M_1 is M_2 -injective and M_2 is M_1 -injective.
- Every automorphism-invariant module satisfies the property C3.
- A CS automorphism-invariant module is quasi-injective.

3. RESULTS

Dickson and Fuller in [5] considered a finite-dimensional algebra R over a field \mathbb{F} with more than two elements and proved that if an indecomposable right R -module M is automorphism-invariant, then M is quasi-injective. They further obtained the following.

Theorem 1. (*Dickson and Fuller, [5]*) *Let R be a finite-dimensional algebra over a field \mathbb{F} with more than two elements. Then the following statements are equivalent;*

- (i) *Each indecomposable right R -module is automorphism-invariant.*
- (ii) *Each indecomposable right R -module is quasi-injective.*
- (iii) *Each indecomposable right R -module has a square-free socle.*

We will provide an example to show that if R is a finite-dimensional algebra over a field \mathbb{F} with two elements, then an indecomposable automorphism-invariant right R -module need not be quasi-injective.

First, note that in an artinian serial ring R , any indecomposable summand of R_R of maximum length is injective. Thus if $T_n(D)$ is the upper triangular matrix ring over a division ring D , then $e_{11}T_n(D)$ is injective and uniserial.

Example. Let $R = \begin{bmatrix} \mathbb{F} & \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} & 0 \\ 0 & 0 & \mathbb{F} \end{bmatrix}$ where \mathbb{F} is a field of order 2.

We know that R is a left serial ring. Note that $e_{11}R$ is a local module, $e_{12}\mathbb{F} \cong e_{22}R$, $e_{13}\mathbb{F} \cong e_{33}R$ and $e_{11}J(R) = e_{12}\mathbb{F} \oplus e_{13}\mathbb{F}$, a direct sum of two minimal right ideals. So the injective hull of $e_{11}R$ is $E(e_{11}R) = E_1 \oplus E_2$, where $E_1 = E(e_{12}\mathbb{F})$ and $E_2 = E(e_{13}\mathbb{F})$.

Now set $A = \text{ann}_r(e_{12}\mathbb{F})$. Then $A = e_{13}\mathbb{F} + e_{33}\mathbb{F}$. Thus $\overline{R} = R/A \cong \begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{bmatrix} = S$. Denote the first row of S by S_1 . It may be checked that S_1 is injective. As \mathbb{F} has only two elements, S_1 has only two endomorphisms, zero and the identity. Take the pre-image L_1 of S_1 in \overline{R} . It is uniserial with composition length 2, and $e_{12}\mathbb{F}$ naturally embeds in L_1 . There is no mapping of $e_{13}\mathbb{F}$ into L_1 . It follows that L_1 is $e_{11}R$ -injective and $e_{12}\mathbb{F}$ -injective. As $e_{22}R \cong e_{12}\mathbb{F}$, L_1 is $e_{22}R$ -injective. There is no map from $e_{33}R$ into L_1 so it is also $e_{33}R$ -injective. Hence L_1 is injective. Thus $E_1 = L_1$ and its ring of endomorphisms has only two elements.

If $B = \text{ann}_r(e_{13}\mathbb{F})$, then $B = e_{12}\mathbb{F} + e_{22}\mathbb{F}$. Thus $R/B \cong \begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{bmatrix}$. The pre-image of S_1 in R/B is L_2 , which is uniserial, and injective. We have $E_2 \cong L_2$ and its ring of endomorphism has only two elements.

Note that $e_{11}R$ has all its composition factors non-isomorphic, both L_1 and L_2 have composition length 2 with $\frac{L_1}{L_1J(R)} \cong \frac{e_{11}R}{e_{11}J(R)}$, $L_1J(R) \cong e_{22}R$, $\frac{L_2}{L_2J(R)} \cong \frac{e_{11}R}{e_{11}J(R)}$, and $L_2J(R) \cong e_{33}R$. Thus L_1, L_2 have isomorphic tops but non-isomorphic socles.

Suppose there exists a non-zero mapping $\sigma : L_1 \rightarrow L_2$. Then $\sigma(L_1) = L_2J(R)$. Thus $\frac{e_{11}R}{e_{11}J(R)} \cong e_{33}R$, which is a contradiction. Therefore, there is no non-zero map between L_1 and L_2 .

Hence the only automorphism of $L_1 \oplus L_2$ is the identity. So $e_{11}R$ is trivially automorphism-invariant but it is not uniform. Then clearly $e_{11}R$ is not quasi-injective as an indecomposable quasi-injective module must be uniform.

Thus, this ring R is an example of a finite-dimensional algebra over a field \mathbb{F} with two elements such that there exists an indecomposable right R -module which is automorphism-invariant but not quasi-injective. \square

Next, we proceed to extend the result of Dickson and Fuller [5] to any right artinian ring. But, first we obtain a useful result on decomposition property of automorphism-invariant modules.

We will show that under certain conditions a decomposition of injective hull $E(M)$ of an automorphism-invariant module M induces a natural decomposition of M .

We will denote the identity automorphism on any module M by I_M .

Lemma 2. *Let M be an automorphism-invariant right module over any ring R . If $E(M) = E_1 \oplus E_2$ and $\pi_1 : E(M) \rightarrow E_1$ is an associated projection, then $M_1 = \pi_1(M)$ is also automorphism-invariant.*

Proof. Let $E(M) = E_1 \oplus E_2$ and $M_1 = \pi_1(M)$, where $\pi_1 : E(M) \rightarrow E_1$ is a projection with E_2 as its kernel. Let σ_1 be an automorphism of E_1 and $x_1 \in M_1$. For some $x \in M$, and $x_2 \in E_2$, we have $x = x_1 + x_2$. Now $\sigma = \sigma_1 \oplus I_{E_2}$ is an automorphism of E . Thus $\sigma(x) = \sigma_1(x_1) + x_2 \in M$, which gives $\sigma_1(x_1) \in M_1$. Hence M_1 is automorphism-invariant. \square

Lemma 3. *Let M be an automorphism-invariant right module over any ring R . Let $E(M) = E_1 \oplus E_2$ such that there exists an automorphism σ_1 of E_1 such that $I_{E_1} - \sigma_1$ is also an automorphism of E_1 . Then*

$$M = (M \cap E_1) \oplus (M \cap E_2).$$

Proof. Set $E = E(M)$. Set $I_E = I_{E_1} \oplus I_{E_2}$, and $\sigma = \sigma_1 \oplus I_{E_2}$. Clearly, both I_E and σ are automorphisms of E . Since M is assumed to be an automorphism-invariant module, M is invariant under automorphisms I_E and σ . Consequently, M is invariant under $I_E - \sigma$ too. Note that $I_E - \sigma = (I_{E_1} - \sigma_1) \oplus 0$. Thus $(I_E - \sigma)(M) = (I_{E_1} - \sigma_1)(M) \subseteq M$. Let $\pi_1 : E \rightarrow E_1$ and $\pi_2 : E \rightarrow E_2$ be the canonical projections. Set $M_1 = \pi_1(M)$ and $M_2 = \pi_2(M)$. Now $M \cap E_1 \subseteq M_1$ and $M \cap E_2 \subseteq M_2$.

Let $0 \neq u_1 \in E_1$. For some $r \in R$, $0 \neq u_1 r \in M$ and thus $u_1 r \in M_1$. Thus $M_1 \subseteq_e E_1$. By Lemma 2, M_1 is automorphism-invariant. Therefore, $M_1 = (I_{E_1} - \sigma_1)^{-1}(M_1)$. Let $x_1 \in M_1$. Then, we have for some $x \in M$, $x = x_1 + x_2$, $x_2 \in E_2$. Now, as $I_{E_1} - \sigma_1$ is an automorphism on E_1 , there exists an element $y_1 \in E_1$ such that $(I_{E_1} - \sigma_1)(y_1) = x_1$, which gives $y_1 \in (I_{E_1} - \sigma_1)^{-1}(M_1) = M_1$. This yields an element $y \in M$ such that $y = y_1 + y_2$ for some $y_2 \in E_2$. We get $(I_E - \sigma)(y) = (I_{E_1} - \sigma_1)(y_1) = x_1$. Thus $x_1 \in (I_E - \sigma)(M)$. As $(I_E - \sigma)(M) \subseteq M$, we get $x_1 \in M$. Hence $M_1 \subseteq M$.

Now, let $u_2 \in M_2$ be an arbitrary element. For some $u_1 \in M_1$, we have $u = u_1 + u_2 \in M$. But we have shown in the previous paragraph that $M_1 \subseteq M$, so $u_1 \in M$. Therefore $u_2 = u - u_1 \in M$. Hence $M_2 \subseteq M$. This gives $M_1 \oplus M_2 \subseteq M$ and hence $M = M_1 \oplus M_2$. Thus $M = (M \cap E_1) \oplus (M \cap E_2)$. \square

A quasi-injective module is obviously automorphism-invariant. In the next result we give a condition under which an automorphism-invariant module must be quasi-injective.

Theorem 4. *Let M be a right module over any ring R such that every summand E_1 of $E(M)$ admits an automorphism σ_1 such that $I_{E_1} - \sigma_1$ is also an automorphism of E_1 , then M is automorphism-invariant if and only if M is quasi-injective.*

Proof. Let M be automorphism-invariant. Set $E = E(M)$. Suppose every summand E_1 of E admits an automorphism σ_1 such that $I_{E_1} - \sigma_1$ is also an automorphism of E_1 .

Let $\sigma \in \text{End}(E)$ be an arbitrary element. Since $\text{End}(E)$ is a clean ring [1], $\sigma = \alpha + \beta$ where α is an idempotent and β is an automorphism.

Let $E_1 = \alpha E$, and $E_2 = (1 - \alpha)E$. Then $E = E_1 \oplus E_2$. By Lemma 3, we have $M = M_1 \oplus M_2$ where $M_1 = M \cap E_1$, $M_2 = M \cap E_2$.

Then clearly $\alpha(M) \subseteq M$. Since M is automorphism-invariant, $\beta(M) \subseteq M$. Thus $\sigma(M) \subseteq M$. Hence M is quasi-injective.

The converse is obvious. \square

As a consequence of this theorem, we may now deduce the following which extends the result of Dickson and Fuller [5] to any algebra (not necessarily finite-dimensional) over a field \mathbb{F} with more than two elements.

Corollary 5. *Let R be any algebra over a field \mathbb{F} with more than two elements. Then the following are equivalent;*

- (i) *Each indecomposable right R -module is automorphism-invariant.*
- (ii) *Each indecomposable right R -module is quasi-injective, that is, R is of right invariant module type.*

Proof. Clearly, for any right R -module E , the multiplication by an element $u \in \mathbb{F}$ where $u \neq 0$ and $u \neq 1$ gives an automorphism σ of E such that $I_E - \sigma$ is also an automorphism of E . Hence the result follows from the above theorem. \square

Corollary 6. ([14]) *Let R be a ring in which 2 is invertible. Then any automorphism-invariant module over R is quasi-injective.*

Proof. Let M be an automorphism-invariant right R -module. Let $E = E(M)$. Let E_1 be any summand of E . We have automorphism $\sigma_1 : E_1 \rightarrow E_1$, given by $\sigma_1(x) = 2x$, $x \in E_1$. Clearly, $I_{E_1} - \sigma_1 = -I_{E_1}$ is also an automorphism of E_1 . By Theorem 4, M is quasi-injective. \square

In the next lemma we give another useful result on decomposition of automorphism-invariant modules.

Lemma 7. *Let M be an automorphism-invariant right module over any ring R . If $E(M) = E_1 \oplus E_2 \oplus E_3$, where $E_1 \cong E_2$, then*

$$M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E_3).$$

Proof. Set $E(M) = E$. Let $E = E_1 \oplus E_2 \oplus E_3$. Let $\sigma : E_1 \rightarrow E_2$ be an isomorphism and let $\pi_1 : E \rightarrow E_1$, $\pi_2 : E \rightarrow E_2$, and $\pi_3 : E \rightarrow E_3$ be the canonical projections. Then $M \cap E_1 \subseteq \pi_1(M)$, $M \cap E_2 \subseteq \pi_2(M)$ and $M \cap E_3 \subseteq \pi_3(M)$.

Let $\eta = \sigma^{-1}$. Consider the map $\lambda_1 : E \rightarrow E$ given by $\lambda_1(x_1, x_2, x_3) = (x_1, \sigma(x_1) + x_2, x_3)$. Clearly, λ_1 is an automorphism of E . Since M is automorphism-invariant, M is invariant under λ_1 and I_E . Consequently, M is invariant under $\lambda_1 - I_E$. Thus $(\lambda_1 - I_E)(M) \subseteq M$. Next, we consider the map $\lambda_2 : E \rightarrow E$ given by $\lambda_2(x_1, x_2, x_3) = (x_1 + \eta(x_2), x_2, x_3)$. This map λ_2 is also an automorphism of E . Thus, as explained above, M is invariant under $\lambda_2 - I_E$ too, that is $(\lambda_2 - I_E)(M) \subseteq M$.

Let $x = (x_1, x_2, x_3) \in M$. Then $(\lambda_1 - I_E)(x) = (0, \sigma(x_1), 0) \in M$. Similarly, we have $(\lambda_2 - I_E)(x) = (\eta(x_2), 0, 0) \in M$. This gives $(\lambda_1 - I_E)(\eta(x_2), 0, 0) = (0, \sigma\eta(x_2), 0) = (0, x_2, 0) \in M$. Thus $\pi_2(M) \subseteq M$. Similarly, $(\lambda_2 - I_E)(0, \sigma(x_1), 0) = (\eta\sigma(x_1), 0, 0) = (x_1, 0, 0) \in M$. Thus $\pi_1(M) \subseteq M$. This yields that $(0, 0, x_3) \in M$, that is, $\pi_3(M) \subseteq M$. This shows that $\pi_1(M) \oplus \pi_2(M) \oplus \pi_3(M) \subseteq M$ and therefore, $M = \pi_1(M) \oplus \pi_2(M) \oplus \pi_3(M)$. Hence $M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E_3)$. \square

As a consequence of the above decomposition, we have the following for socle of an indecomposable automorphism-invariant module.

Corollary 8. *If M is an indecomposable automorphism-invariant right module over any ring R , then $\text{Soc}(M)$ is square-free.*

Proof. Let M be an indecomposable automorphism-invariant module. Suppose M has two isomorphic simple submodules S_1 and S_2 . Then $E(M) = E_1 \oplus E_2 \oplus E_3$, where $E_1 = E(S_1)$, $E_2 = E(S_2)$ and $E_1 \cong E_2$. By Lemma 7, M decomposes as $M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E_3)$, a contradiction to our assumption that M is indecomposable. Hence $\text{Soc}(M)$ is square-free. \square

Next, we have the following for any indecomposable semi-artinian automorphism-invariant module.

Corollary 9. *Let R be any ring and let M be any indecomposable semi-artinian automorphism-invariant right R -module. Then one of the following statements holds:*

- (i) M is uniform and quasi-injective.
- (ii) Any simple submodule S of M has identity as its only automorphism.

Proof. Let M be an indecomposable semi-artinian automorphism-invariant right R -module. Since M is semi-artinian, $\text{Soc}(M) \neq 0$. By Corollary 8, we know that $\text{Soc}(M)$ is square-free. Suppose S is a simple submodule of M . Now $D = \text{End}(S)$ is a division ring.

Suppose $|D| > 2$. Then there exists a $\sigma \in D$ such that $\sigma \neq 0$ and $\sigma \neq I_S$. Then $I_S - \sigma$ is an automorphism of S . Let $E = E(M)$ and $E_1 = E(S) \subseteq E$. Then $E = E_1 \oplus E_2$ for some submodule E_2 of E . Let $\sigma_1 \in \text{End}(E_1)$ be an extension of σ . Then σ_1 is an automorphism of E_1 and $(I_{E_1} - \sigma_1)(S) = (I_S - \sigma)(S) \neq 0$. Hence $I_{E_1} - \sigma_1$ is an automorphism of E_1 . Thus, by Lemma 3, $M = (M \cap E_1) \oplus (M \cap E_2)$. As M is indecomposable, we must have $M = M \cap E_1$. Therefore, M is uniform. Then $\text{End}(E(M))$ is a local ring. Therefore for any $\alpha \in \text{End}(E(M))$, α is an automorphism or $I - \alpha$ is an automorphism. In any case $\alpha(M) \subseteq M$. Therefore M is quasi-injective.

Now, if M is not uniform then $|D| = 2$, that is $D = \text{End}(S) \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$. In this case, the only automorphism of S is the identity automorphism. \square

Remark 10. Recall that an algebra A is said to be of finite module type if A has only a finite number of non-isomorphic indecomposable right modules. In regard to Corollary 8, we would like to mention here that Curtis and Jans proved that if A is an algebra over an algebraically closed field \mathbb{F} such that for each indecomposable right A -module M , $\text{Soc}(M)$ is square-free, then A is of finite module type (see [4]). This was extended by Dickson and Fuller who proved that if A is an algebra over any field \mathbb{F} such that A is of right invariant module type then A has finite module type [5].

We call a ring R to be of *right automorphism-invariant type* (in short, RAI-type), if every finitely generated indecomposable right R -module is automorphism-invariant. We would like to understand the structure of right artinian rings of RAI-type.

Lemma 11. Let R be a right artinian ring of RAI-type. Let $e \in R$ be an indecomposable idempotent such that eR is not uniform. Let A be a right ideal contained in $\text{Soc}(eR)$. Then $\text{Soc}(eR) = A \oplus A'$ where A' has no simple submodule isomorphic to a simple submodule of A and eR/A' is quasi-projective.

Proof. As $\text{Soc}(eR)$ is square-free, $\text{Soc}(eR) = A \oplus A'$ where A' has no simple submodule isomorphic to a simple submodule of A . If for some $ere \in eRe$, $ereA' \not\subseteq A'$, then for some minimal right ideal $S \subset A'$, $ereS \not\subseteq A'$. This gives that S is isomorphic to a simple submodule contained in A , a contradiction. Hence eR/A' is quasi-projective. \square

Lemma 12. Let R be a right artinian ring of RAI-type. Then any uniserial right R -module is quasi-projective.

Proof. Let A be a uniserial right R -module with composition length $l(A) = n \geq 2$. We will prove the result by induction. Suppose first that $n = 2$. In this case, we can take $J(R)^2 = 0$. For some indecomposable idempotent $e \in R$, we have $A \cong eR/B$ for some $B \subseteq \text{Soc}(eR)$. By Lemma 11, A is quasi-projective.

Now consider $n > 2$ and assume that the result holds for $n - 1$. Let $0 \neq \sigma : A \rightarrow A/C$ be a homomorphism where $C \neq 0$. Suppose σ cannot be lifted to a homomorphism $\eta : A \rightarrow A$. Let $F = \text{Soc}(A)$. Then $F \subseteq \text{Ker}(\sigma)$. We get a mapping $\bar{\sigma} : \frac{A}{F} \rightarrow \frac{A}{C}$. By the induction hypothesis, there exists a homomorphism $\bar{\eta} : \frac{A}{F} \rightarrow \frac{A}{F}$ such that $\bar{\sigma} = \pi \bar{\eta}$, where $\pi : \frac{A}{F} \rightarrow \frac{A}{C}$ is a natural homomorphism.

Let $M = A \times A$, and $N = \{(a, b) \in M : \bar{\eta}(a+F) = b+F\}$. Then N is a submodule of M . Now there exist elements $x \in A$ and indecomposable idempotent $e \in R$ such that $A = xR$ and $xe = x$. Fix an element $y \in A$ such that $\bar{\eta}(x+F) = y+F$ and $ye = y$. Set $z = (x, y)$. Then $z \in N$ and $N_1 = zR$ is local. Let π_1, π_2 be the associated projections of M onto the first and second components of M , respectively. Then $\pi_1(N_1) = A$.

Now, we claim that N_1 is uniserial. If N_1 is not uniform, then $\text{Soc}(N_1) = \text{Soc}(M)$. Therefore $\text{Soc}(N_1)$ is not square-free, which is a contradiction by Lemma 8. Thus N_1 is uniform. It follows that N_1 embeds in A under π_1 or π_2 . Hence N_1 is uniserial. As $\pi_1(N_1) = A$, and $l(N_1) \leq l(A)$, it follows that $\pi_1|_{N_1}$ is an isomorphism. Thus given any $x \in A$, there exists a unique $y \in A$ such that $(x, y) \in N_1$. We get a homomorphism $\lambda : A \rightarrow A$ such that $\lambda(x) = y$ if and only if $(x, y) \in N_1$. Clearly λ lifts $\bar{\eta}$ and hence it also lifts σ . This proves that A is quasi-projective. \square

Lemma 13. *Let R be a right artinian ring of RAI-type. Let A_R be any uniserial module. Then the rings of endomorphisms of different composition factors of A are isomorphic.*

Proof. Let A be a uniserial right R -module with $l(A) = 2$. Let $C = \text{ann}_r(A)$ and $\overline{R} = R/C$. As A_R is quasi-projective, A is a projective \overline{R} -module. Thus there exists an indecomposable idempotent $e \in R$ such that $A \cong \overline{eR}$. As \overline{R} embeds in a finite direct sum of copies of A , there exists an indecomposable idempotent $f \in R$ such that $\text{Soc}(A) \cong \frac{fR}{fJ(R)}$, $\overline{eJ(R)} = \overline{exfR}$ for some $x \in J(R)$. We get an embedding $\sigma : \frac{eRe}{eJ(R)e} \rightarrow \frac{fRf}{fJ(R)f}$ defined as $\sigma(ere + eJ(R)e) = fr'f + fJ(R)f$ whenever $\overline{ereexf} = \overline{exffr'f}$; $ere \in eRe, fr'f \in fRf$. Let $z = fvf \in fRf$. We get an \overline{R} -homomorphism $\eta : \overline{eJ(R)} \rightarrow \overline{eJ(R)}$ such that $\eta(\overline{exf}) = \overline{exffvf}$. As \overline{eR} is quasi-injective, there exists an \overline{R} -homomorphism $\lambda : \overline{eR} \rightarrow \overline{eR}$ extending η . Now $\lambda(\overline{e}) = \overline{ere}$ for some $r \in R$. Then $\overline{ereexf} = \lambda(\overline{exf}) = \eta(\overline{exf}) = \overline{exffvf}$, which gives that σ is onto. Hence $\frac{eRe}{eJ(R)e} \cong \frac{fRf}{fJ(R)f}$. Thus the result holds whenever $l(A) = 2$. If $l(A) = n > 2$, the result follows by induction on n . \square

Lemma 14. *Let R be a right artinian ring of RAI-type. Then we have the following.*

- (i) *Let D be a division ring and $x \in R$. Let xR be a local module such that for any simple submodule S of $\text{Soc}(xR)$, $D = \text{End}(S)$. Then $\text{End}(xR/xJ(R)) \cong D$.*
- (ii) *Let xR be a local module and $D = \text{End}(xR/xJ(R))$ where $x \in R$. Then $\text{End}(S) \cong D$ for every composition factor S of xR .*
- (iii) *Let xR, yR be two local modules where $x, y \in R$. If $\text{End}(xR/xJ(R)) \not\cong \text{End}(yR/yJ(R))$, then $\text{Hom}(xR, yR) = 0$.*

Proof. (i) There exists an $n \geq 1$ such that $xJ(R)^n = 0$ but $xJ(R)^{n-1} \neq 0$. If $n = 1$, then xR is simple, so the result holds. We apply induction on n . Suppose $n > 1$ and assume that the result holds for $n - 1$. Now $xJ(R)J(R)^{n-1} = 0$, but $xJ(R)J(R)^{n-2} \neq 0$. Therefore there exists an element $y \in xJ(R)$ such that yR is local and $yJ(R)^{n-1} = 0$ but $yJ(R)^{n-2} \neq 0$. By the induction hypothesis, $\text{End}(yR/yJ(R)) \cong D$. In fact, for any simple submodule S' of $eJ(R)/xJ(R)^2$, $\text{End}(S') \cong D$. Consider the local module $M = xR/xJ(R)^2$. Let S' be a simple submodule of M . Then $\text{Soc}(M) = S' \oplus B$ for some $B \subset \text{Soc}(M)$. Then $\text{End}(S') \cong D$. As $A = M/B$ is uniserial, $\text{Soc}(A) \cong S'$ and $A/AJ(R) \cong xR/xJ(R)$. By Lemma 13, $\text{End}(xR/xJ(R)) \cong D$.

- (ii) Let S be a simple submodule of $\text{Soc}(xR)$ and B be a complement of S in xR . Then $\overline{xR} = xR/B$ is uniform and $\text{Soc}(\overline{xR}) \cong S$. By (i), $\text{End}(S) \cong \text{End}(\frac{\overline{xR}}{xJ(R)}) \cong \text{End}(xR/xJ(R)) = D$. Hence $\text{End}(S) \cong D$ for any simple submodule S of xR . Let S_1 be any composition factor of xR . Then there exists a local submodule yR of xR such that $S_1 \cong yR/yJ(R)$. By (i), $\text{End}(S_1) \cong \text{End}(S) \cong D$, where S is a simple submodule of yR .

- (iii) It is immediate from (ii). \square

Now, we are ready to give the structure of indecomposable right artinian rings of RAI-type.

Theorem 15. *Let R be an indecomposable right artinian ring of RAI-type. Then the following hold.*

- (i) *There exists a division ring D such that $\text{End}(S) \cong D$ for any simple right R -module S . In particular, $R/J(R)$ is a direct sum of matrix rings over D .*
- (ii) *If $D \not\cong \mathbb{Z}/2\mathbb{Z}$, then every finitely generated indecomposable right R -module is quasi-injective. In this case, R is right serial.*

Proof. (i) Let $e \in R$ be an indecomposable idempotent and $D = eRe/eJ(R)e$. By above lemma, every composition factor S of eR satisfies $\text{End}(S) \cong D$. Now $R_R = \oplus_{i=1}^n e_i R$ where e_i are orthogonal indecomposable idempotents with $e_1 = e$. Let A be the direct sum of those $e_j R$ for which $\frac{e_j R e_j}{e_j J(R) e_j} \cong D$. Consider any e_k for which $\frac{e_k R e_k}{e_k J(R) e_k} \not\cong D$. It follows from Lemma 14(iii) that $A e_k R = 0 = e_k R A$. Consequently, $A = e_k R$ and we get that $R = A \oplus B$ for some ideal B . As R is indecomposable, we get $R = A$. This proves (i).

- (ii) Suppose $D \not\cong \mathbb{Z}/2\mathbb{Z}$. It follows from Corollary 9 that every indecomposable right R -module is uniform and quasi-injective. In particular, if $e \in R$ is an indecomposable idempotent, then any homomorphic image of eR is uniform, which gives that eR is uniserial. Hence R is right serial. \square

Theorem 16. [16] *Let R be a right artinian ring such that $J(R)^2 = 0$. If every finitely generated indecomposable right R -module is local, then R satisfies the following conditions.*

- (a) *Every uniform right R -module is either simple or is injective with composition length 2.*
- (b) *R is a left serial ring.*
- (c) *For any indecomposable idempotent $e \in R$ either $eJ(R)$ is homogeneous or $l(eJ(R)) \leq 2$.*

Conversely, if R satisfies (a), (b), (c) and $l(eJ(R)) \leq 2$, then every finitely generated indecomposable right R -module is local.

Example. Let $R = \begin{bmatrix} \mathbb{F} & \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} & 0 \\ 0 & 0 & \mathbb{F} \end{bmatrix}$ where $\mathbb{F} = \frac{\mathbb{Z}}{2\mathbb{Z}}$.

Then R is a left serial ring. We have already seen that $e_{11}R$ is an indecomposable module which is automorphism-invariant but not quasi-injective. It follows from Theorem 16 that every finitely generated indecomposable right R -module is local. Thus the only indecomposable modules which are not simple are the homomorphic images of $e_{11}R$, which are $e_{11}R$, $\frac{e_{11}R}{e_{12}\mathbb{F}}$, and $\frac{e_{11}R}{e_{13}\mathbb{F}}$. These are all automorphism-invariant. It follows from Theorem 16 that any finitely generated indecomposable right R -module is local. Thus this ring R is an example of a ring where every finitely generated indecomposable right R -module is automorphism-invariant. \square

Example. Let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ and $R = \begin{bmatrix} \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} & 0 & 0 \\ 0 & 0 & \mathbb{F} & 0 \\ 0 & 0 & 0 & \mathbb{F} \end{bmatrix}$.

This ring R is left serial and $J(R)^2 = 0$. Now $e_{11}J(R) = e_{12}\mathbb{F} \oplus e_{13}\mathbb{F} \oplus e_{14}\mathbb{F}$, a direct sum of non-isomorphic minimal right ideals. It follows from condition (c) in Theorem 16 that there exists a finitely generated indecomposable right R -module that is not local. We have $E_1 = E(e_{12}\mathbb{F})$, $E_2 = E(e_{13}\mathbb{F})$, $E_3 = E(e_{14}\mathbb{F})$, each of them has composition length 2. Now $e_{11}R$ has two homomorphic images $A_1 = \frac{e_{11}R}{e_{14}\mathbb{F}}$ and $A_2 = \frac{e_{11}R}{e_{12}\mathbb{F}}$ such that $\text{Soc}(A_1) \cong e_{12}\mathbb{F} \oplus e_{13}\mathbb{F}$ and $\text{Soc}(A_2) \cong e_{13}\mathbb{F} \oplus e_{14}\mathbb{F}$. So we get $B_1 \subseteq E_1 \oplus E_2 \subseteq E_1 \oplus E_2 \oplus E_3$ such that $A_1 \cong B_1$. Similarly, we have $A_2 \cong B_2 \subseteq E_2 \oplus E_3$. Let $E = E_1 \oplus E_2 \oplus E_3$. Its only automorphism is I_E . Thus any essential submodule of E is automorphism-invariant. Now $B = B_1 + B_2 \subseteq_e E$, so B is automorphism-invariant and B is not local. We prove that B is indecomposable. We have $B_1 \cap B_2 = e_{13}\mathbb{F}$. Notice that any submodule of $E_1 \oplus E_2$ that is indecomposable and not uniserial is B_1 . Suppose a simple submodule S of B is a summand of B . But $S \subset B_1$ or $S \subset B_2$, therefore B_1 or B_2 decomposes, which is a contradiction. As $l(B) = 5$, B has a summand C_1 with $l(C_1) = 2$. Then C_1 being uniserial, it equals one of E_i .

Case 1. $C_1 = E_1$. Then $B = C_1 \oplus C_2$, where $\text{Soc}(C_2) \cong B_2$. As C_2 has no uniserial submodule of length 2, the projection of B_1 in C_2 equals $\text{Soc}(C_2)$, we get B_1 is semi-simple, which is a contradiction.

Similarly other cases follow. Hence B is indecomposable. \square

Now, we proceed to answer the question of Lee and Zhou [14] whether every automorphism-invariant module is pseudo-injective in the affirmative for modules with finite Goldie dimension.

Theorem 17. *If M is an automorphism-invariant module with finite Goldie dimension, then M is pseudo-injective.*

Proof. Let N be a submodule of M . Let $\sigma : N \rightarrow M$ be a monomorphism. Then σ can be extended to a monomorphism $\sigma' : E(N) \rightarrow E(M)$. Now, we may write $E(M) = \sigma'(E(N)) \oplus P = E(N) \oplus Q$ for some submodules P and Q of $E(M)$. Note that $\sigma'(E(N)) \cong E(N)$. Since M has finite Goldie dimension, $E(M)$ has finite Goldie dimension. Thus $E(M)$ is a directly-finite injective module, and hence $E(M)$ satisfies internal cancellation property. Therefore, $P \cong Q$. Thus, there exists an isomorphism $\varphi : Q \rightarrow P$. Now consider the mapping $\lambda : E(M) \rightarrow E(M)$ defined as $\lambda(x+y) = \sigma'(x) + \varphi(y)$ where $x \in E(N)$ and $y \in Q$. Clearly, λ is an automorphism of $E(M)$. Since M is assumed to be automorphism-invariant, we have $\lambda(M) \subseteq M$. Thus $\lambda|_M : M \rightarrow M$ extends σ . This shows that M is pseudo-injective. \square

It is well known that if R is a ring such that each cyclic right R -module is injective then R is semisimple artinian. For more details on rings characterized by properties of their cyclic modules, the reader is referred to [9]. We would like to understand the structure of rings for which each cyclic module is automorphism-invariant. In [14] it is shown that if every 2-generated right module over a ring R is automorphism-invariant, then R is semisimple artinian.

A ring R is called a *right SI ring* if every singular right R -module is injective [6]. In [8] Huynh, Jain, and López-Permouth proved that a simple ring R is a right SI ring if and only if every cyclic singular right R -module is π -injective. Their proof can be adapted to show that a simple right noetherian ring R is a right SI ring if and only if every cyclic singular right R -module is automorphism-invariant.

The following lemma due to Huynh et al [7, Lemma 3.1] is crucial for proving our result. This lemma is, in fact, a generalization of a result of J. T. Stafford given in [2, Theorem 14.1].

Lemma 18. ([7]) *Let R be a simple right Goldie ring which is not artinian and M a torsion right R -module. If $M = aR + bR$, where bR is of finite composition length and f is a non-zero element of R then $M = (a + bxf)R$ for some $x \in R$.*

We know that for a prime right Goldie ring R , singular right R -modules are the same as torsion right R -modules. Now, we are ready to prove the following.

Theorem 19. *Let R be a simple right noetherian ring. Then R is a right SI ring if and only if every cyclic singular right R -module is automorphism-invariant.*

Proof. Let R be a simple right noetherian ring such that every cyclic singular right R -module is automorphism-invariant. We wish to show that R is a right SI ring. If $\text{Soc}(R_R) \neq 0$, then as R is a simple ring, $R = \text{Soc}(R_R)$ and hence R is simple artinian.

Now, assume $\text{Soc}(R_R) = 0$. Let M be any artinian right R -module. Since any module is homomorphic image of a free module, we may write $M \cong F/K$ where F is a free right R -module. We first claim that K is an essential submodule of F . Assume to the contrary that K is not essential in F . Let T be a complement of K in F . Note that $M \cong \frac{F}{K} \supset \frac{K \oplus T}{K} \cong T$. Since M is an artinian module, $\text{Soc}(M) \neq 0$ and consequently $\text{Soc}(T) \neq 0$. This yields that $\text{Soc}(F) \neq 0$, a contradiction to the assumption that $\text{Soc}(R_R) = 0$. Therefore, K is an essential submodule of F and hence M is a singular module. Let C be a cyclic submodule of M . We have $\text{Soc}(C) \neq 0$. As R is right noetherian and C is a cyclic right R -module, C is noetherian. Thus we have $\text{Soc}(C) = \bigoplus_{i=1}^k S_i$ where each S_i is a simple right R -module. By the above lemma, it follows that $C \oplus S_1$ is cyclic. By induction, it may be shown that $C \oplus \text{Soc}(C)$ is cyclic. Now as $C \oplus \text{Soc}(C)$ is a cyclic singular right R -module, by assumption $C \oplus \text{Soc}(C)$ is automorphism-invariant. Hence $\text{Soc}(C)$ is C -injective. Therefore, $\text{Soc}(C)$ splits in C and hence $C = \text{Soc}(C) \subset M$. Thus M is semisimple. This shows that any artinian right R -module M is semisimple.

Now, we prove that every singular module over R is semisimple, or equivalently, for each essential right ideal E of R , R/E is semisimple. By the above claim, it suffices to show that R/E is artinian. Set $N = R/E$. If N is not artinian, then we get $0 \subset V_1 \subset N$ with V_1 not artinian. Now N is torsion, so is V_1 . Therefore, $Q = N \oplus V_1$ is torsion and hence cyclic by Lemma 18. Thus we can write $xR = N \oplus V_1$ for some $x \in R$. By the assumption, xR is automorphism-invariant. Hence V_1 is N -injective. So $N = N_1 \oplus V_1$. Repeat the process with V_1 , so $V_1 = N_2 \oplus V_2$, where $N_2 \neq 0$ and V_2 is not artinian. Continuing this process, we get an infinite direct sum of N_i in N , which is a contradiction. Thus we conclude that any singular right R -module is artinian and consequently semisimple.

Thus R is a right nonsingular ring such that every singular right R -module is semisimple. Hence, by [6], R is a right SI ring.

The converse is obvious. \square

4. QUESTIONS

Question 1: Does every automorphism-invariant module satisfy the property C2 ?

Lee and Zhou [14] have shown that every automorphism-invariant module satisfies the property C3.

Question 2: What is example of an automorphism-invariant module which is not pseudo-injective?

In Theorem 17 above, we have shown that such a module cannot have finite Goldie dimension.

A module M is called skew-injective if for every submodule N of M , any endomorphism of N extends to an endomorphism of M . In [9] it is asked whether every skew-injective module with essential socle is quasi-injective. We ask the following

Question 3: Is every automorphism-invariant module with essential socle a quasi-injective module?

Call a ring R to be a *right a-ring* if each right ideal of R is automorphism-invariant.

Question 4: Describe the structure of a right a-ring.

Call a ring R to be a *right Σ -a-ring* if each right ideal of R is a finite direct sum of automorphism-invariant right ideals.

Question 5: Describe the structure of a right Σ -a-ring.

Question 6: Let R be a simple ring such that R_R is automorphism-invariant. Must R be a right self-injective ring ?

In fact, this question is open even when R_R is pseudo-injective (see [3]).

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